Chapter 10
Unsteady State Heat Transfer

Theory

The governing equation for unsteady state heat transfer in rectangular coordinates in the x-direction is:

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

where

\[ T = \text{temperature, } ^\circ\text{C} \]
\[ t = \text{time, s} \]
\[ x = \text{distance from the center plane, m} \]
\[ \alpha = \text{thermal diffusivity, } m^2/s \]

Analytical solutions for various initial and boundary conditions are available in the literature (e.g., Ref. 5 and 6). The solution is of the form:

\[ \frac{T_e - T}{T_e - T_o} = \sum_n A_n \exp(B_n Fo) \]

where

\[ T_e = \text{equilibrium temperature (temperature of the environment)} \]
\[ T_o = \text{initial temperature} \]
\[ T = \text{temperature at time } t \text{ and point } x \]
\[ A_n = \text{constant} \]
\[ B_n = \text{constant} \]
\[ Fo = \text{Fourier number } (=\alpha t/L^2) \]
Table 10.1 Solutions of the unsteady state heat transfer equation with uniform initial temperature and constant surface temperature (negligible surface resistance, \( \text{Bi} > 40 \))

1) Local Temperature

a) Rectangular coordinates

\[
\frac{T_e - T}{T_e - T_0} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \cos \left( \frac{(2n + 1)\pi x}{2L} \right) \exp \left( -\frac{(2n + 1)^2\pi^2}{4} \text{Fo} \right)
\]

b) Cylindrical coordinates

\[
\frac{T_e - T}{T_e - T_0} = 2 \sum_{n=1}^{\infty} \frac{J_0(\delta_n^2)}{\delta_n J_1(\delta_n R)} \exp \left( -R^2\delta_n^2 \text{Fo} \right)
\]

with \( \delta_n \) being roots of the Bessel function \( J_0(\delta_n R) = 0 \).

The first five roots of \( J_0(x) \) are: 2.4048, 5.5201, 8.6537, 11.7915 and 14.9309 (see Table A.1). Consequently,

\( \delta_1 = 2.4048/R \), \( \delta_2 = 5.5201/R \), \( \delta_3 = 8.6537/R \), \( \delta_4 = 11.7915/R \) and \( \delta_5 = 14.9309/R \)

b) Spherical coordinates

\[
\frac{T_e - T}{T_e - T_0} = -\frac{2R}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \left( \frac{n\pi r}{R} \right) \exp \left( -n^2\pi^2 \text{Fo} \right)
\]

2) Mean Temperature

a) Rectangular coordinates

\[
\frac{T_e - T_m}{T_e - T_0} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} \exp \left( -\frac{(2n + 1)^2\pi^2}{4} \text{Fo} \right) = \frac{8}{\pi^2} \left( \exp(-2.47 \text{Fo}) + \frac{1}{9} \exp(-22.2 \text{Fo}) + \frac{1}{25} \exp(-61.7 \text{Fo}) + \ldots \right)
\]

b) Cylindrical coordinates

\[
\frac{T_e - T_m}{T_e - T_0} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\pi^2}{R^2\delta_n^4} \exp \left( -R^2\delta_n^2 \text{Fo} \right) = \frac{4}{\pi^2} \left( 1.7066 \exp(-5.783 \text{Fo}) + 0.324 \exp(-30.5 \text{Fo}) + 0.132 \exp(-74.9 \text{Fo}) + \ldots \right)
\]

c) Spherical coordinates

\[
\frac{T_e - T_m}{T_e - T_0} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left( -n^2\pi^2 \text{Fo} \right) = \frac{6}{\pi^2} \left( \exp(-\pi^2 \text{Fo}) + \frac{1}{4} \exp(-4\pi^2 \text{Fo}) + \frac{1}{9} \exp(-9\pi^2 \text{Fo}) + \ldots \right)
\]

where

\( A \) = heat transfer surface area, \( m^2 \)
\( \alpha \) = thermal diffusivity, \( m^2/s \)
\( \text{Bi} \) = Biot number \( (= hR/k) \)
\( c_p \) = heat capacity, \( J/kg \degree C \)
\( \text{erf} \) = the error function
erfc = the complementary error function defined as erfc = 1 - erf
Fo = Fourier number (\( = \alpha t/L^2 \))
h = heat transfer coefficient, W/m^2 °C
\( J_0 \) = Bessel function of the first kind of order zero
\( J_1 \) = Bessel function of the first kind of order one
k = thermal conductivity, W/m^2°C
L = half thickness of the plate, m
m = mass, kg

Table 10.2 Solutions of the unsteady state heat transfer equation for uniform initial temperature and both resistances, surface and internal, significant \((0.1 < Bi < 40)\)

1) Local Temperature
a) Rectangular coordinates
\[
\frac{T_e - T}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{2Bi}{\delta_n^2 + Bi^2 + Bi \cos(\delta_n)} \exp(-\delta_n^2 Fo) \\
\text{with} \quad \delta_n \text{ roots of: } \delta \tan\delta = Bi
\]
b) Cylindrical coordinates
\[
\frac{T_e - T}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{2Bi}{\delta_n^2 + Bi^2 + Bi J_0(\delta_n)} \exp(-\delta_n^2 Fo) \\
\text{with} \quad \delta_n \text{ roots of: } \delta J_1(\delta) = Bi J_0(\delta)
\]
c) Spherical coordinates
\[
\frac{T_e - T}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{2Bi}{\delta_n^2 + Bi^2 + Bi \sin(\delta_n)} \exp(-\delta_n^2 Fo) \\
\text{with} \quad \delta_n \text{ roots of: } \delta \cot\delta = 1 - Bi
\]

For the center of the sphere \((r = 0)\)
\[
\frac{T_e - T}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{2Bi}{\delta_n^2 + Bi^2 - Bi \sin(\delta_n)} \exp(-\delta_n^2 Fo)
\]

2) Mean Temperature
a) Rectangular coordinates
\[
\frac{T_e - T_m}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{2Bi^2}{(\delta_n^2 + Bi^2 + Bi) \delta_n^2} \exp(-\delta_n^2 Fo) \\
\text{with} \quad \delta_n \text{ roots of: } \delta \tan\delta = Bi
\]
b) Cylindrical coordinates
\[
\frac{T_e - T_m}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{4Bi^2}{(\delta_n^2 + Bi^2) \delta_n^2} \exp(-\delta_n^2 Fo) \\
\text{with} \quad \delta_n \text{ roots of: } \delta J_1(\delta) = Bi J_0(\delta)
\]
c) Spherical coordinates
\[
\frac{T_e - T_m}{T_e - T_0} = \sum_{n=1}^{\infty} \frac{6Bi^2}{(\delta_n^2 + Bi^2 - Bi) \delta_n^2} \exp(-\delta_n^2 Fo) \\
\text{with} \quad \delta_n \text{ roots of: } \delta \cot\delta = 1 - Bi
\]

Table 10.3 Unsteady state heat transfer with negligible internal resistance \((0.1 < Bi)\)
\[
\frac{T_e - T}{T_e - T_0} = \exp\left(-\frac{hA}{c_p m} t\right) = \exp(-Bi Fo)
\]
\[
t = \frac{mc_p}{hA} \ln \frac{T_e - T_0}{T_e - T}
\]
Table 10.4 Solutions of the unsteady state heat transfer equation for a semi-infinite body with uniform initial temperature

1) Constant surface temperature

\[ \frac{T_e - T}{T_e - T_o} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \]

2) Convection at the surface

\[ \frac{T_e - T}{T_e - T_o} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \left[ \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right) \right] \left[ \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right) \right] \]

where

- \( R \) = radius of cylinder or sphere, m
- \( r \) = distance of the point from the centerline of the cylinder or center of the sphere, m
- \( x \) = distance of the point from the center plane of the plate, or from the surface in the case of semi-infinite body, m
- \( t \) = time, s
- \( T_e \) = equilibrium temperature (temperature of the environment or surface temperature)
- \( T_o \) = initial temperature
- \( T \) = temperature at time \( t \) and point \( x \)
- \( T_m \) = mean temperature

To calculate the temperature as a function of time and position in a solid:

1. Identify the geometry of the system. Determine if the solid can be considered as plate, infinite cylinder, or sphere.
2. Determine if the surface temperature is constant. If not, calculate the Biot number and decide the relative importance of internal and external resistance to heat transfer.
3. Select the appropriate equation.
4. Calculate the Fourier number.
5. Find the temperature by applying the selected equation (if \( F_0 > 0.2 \), calculate the temperature either using only the first term of the series solution or using Heisler or Gurnie-Lurie charts).

**REVIEW QUESTIONS**

Which of the following statements are true and which are false?

1. If the temperature at any given point of a body changes with time, unsteady state heat transfer occurs.
2. Thermal diffusivity is a measure of the ability of a material to transfer thermal energy by conduction compared to the ability of the material to store thermal energy.
3. Materials with high thermal diffusivity will need more time to reach equilibrium with their surroundings.

4. The Biot number (Bi) expresses the relative importance of the thermal resistance of a body to that of the convection resistance at its surface.

5. If Bi, the external resistance is negligible.

6. If Bi > 40, the surface temperature may be assumed to be equal to the temperature of the surroundings.

7. If Bi, the internal resistance is significant.

8. If Bi Bi, the temperature of the body may be assumed to be uniform.

9. Problems with Bi are treated with the lumped capacitance method.

10. The Fourier number (Fo) has dimensions of time.

11. The Heisler charts give the temperature at the center of an infinite slab, infinite cylinder, and sphere when Fo > 0.2.

12. The Gurney-Lurie charts give the temperature at any point of an infinite slab, infinite cylinder, and sphere when Fo > 0.2.

13. Thermal penetration depth is defined as the distance from the surface at which the temperature has changed by 10% of the initial temperature difference.

14. Until the thermal penetration depth becomes equal to the thickness of a finite body heated from one side, the body can be treated as a semi-infinite body.

15. A cylinder of finite length can be treated as an infinitely long cylinder if the two ends of the cylinder are insulated or if its length is at least 10 times its diameter.

Examples

Example 10.1

A steak 2 cm thick is put on a hot metallic plate. The surface of the steak in contact with the hot surface immediately attains a temperature of 120°C and retains this temperature. Calculate the temperature 1.1 cm from the hot surface of the steak after 15 min, if the initial temperature of the meat is 5°C and the thermal diffusivity of the meat is 1.4 × 10⁻⁷ m²/s.

Solution

Step 1
Draw the process diagram:
Step 2
Identify the geometry of the system.
The shape of the steak is a plate heated from one side.

Step 3
Examine the surface temperature.
The surface temperature is constant.

Step 4
Select the appropriate equation.
The equation for a plate with uniform initial temperature and constant surface temperature is (Table 10.1):

\[
\frac{T_e - T}{T_e - T_o} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \left( \frac{(2n + 1)\pi x}{2L} \right) \exp \left( -\frac{(2n + 1)^2 \pi^2}{4} \frac{\alpha t}{L^2} \right)
\] (10.1)

Step 5
Calculate the Fourier number.
Since the steak is heated only from one side, the characteristic dimension is the thickness of the steak, not the half thickness. Therefore:

\[
Fo = \frac{\alpha t}{L^2} = \frac{(1.4 \times 10^{-7} \text{ m}^2/\text{s})(15 \times 60)}{0.02^2 \text{ m}^2} = 0.315
\]

Step 6
Calculate the temperature
Since \( F_o > 0.2 \), only the first term of the sum in eqn (10.1) can be used without appreciable error. Alternatively, the solution can be read directly from a Gurney-Lurie chart (see for example Ref. 3), which gives the dimensionless ratio \( (T_e - T)/(T_e - T_o) \) vs. the Fourier number. The temperature below is calculated both ways:

i) From the equation:

\[
\frac{T_e - T}{T_e - T_o} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \left( \frac{(2n + 1)\pi x}{2L} \right) \exp \left( -\frac{(2n + 1)^2 \pi^2}{4} \frac{\alpha t}{L^2} \right)
\]  
\[
\approx \frac{4}{\pi} \frac{(-1)^0}{2 \times 0 + 1} \cos \left( \frac{2 \times 0 + 1)\pi x}{2L} \right) \exp \left( -\frac{(2 \times 0 + 1)^2 \pi^2}{4} \frac{\alpha t}{L^2} \right)
\]  
\[
= \frac{4}{\pi} \cos \left( \frac{\pi x}{2L} \right) \exp \left( -\frac{\pi^2}{4} \frac{\alpha t}{L^2} \right) = \frac{4}{\pi} \cos \left( \frac{\pi \times 0.009}{2 \times 0.02} \right) \exp \left( -\frac{\pi^2}{4} 0.315 \right)
\]
\[
= 0.445
\]

Calculate the temperature as:

\[
\frac{T_e - T}{T_e - T_o} = \frac{120 - T}{120 - 5} = 0.445 \rightarrow T = 68.8^\circ \text{C}
\]
ii) From the Gurnie-Lurie chart:

a) Find the value of $\text{Fo} = 0.315$ on the x-axis of the Gurney-Lurie chart.

b) Find the curve with $\frac{k}{hL} = 0$ (constant surface temperature or $\text{Bi} \to \infty$) and $\frac{x}{L} = 0.009/0.02 = 0.45$.

c) Read the dimensionless temperature on the y-axis as 
\[ \frac{\text{T}_e - \text{T}}{\text{T}_e - \text{T}_o} = 0.45. \]

d) Calculate the temperature as 
\[ \frac{\text{T}_e - \text{T}}{\text{T}_e - \text{T}_o} = \frac{120 - \text{T}}{120 - 5} = 0.45 \to \text{T} = 68.3^\circ\text{C} \]

Comment: The reading from the chart can only be approximate

Example 10.2

A hot dog 1.5 cm in diameter and 16 cm in length with 5°C initial temperature is immersed in boiling water. Calculate a) the temperature 3 mm under the surface after 2 min, b) the temperature at the center after 2 min, c) the time necessary to reach 81°C at the center, d) the average temperature of the hot dog after 2 min, and e) the heat that will be transferred to the hot dog during the 2 min of boiling. Assume that the thermal conductivity is 0.5W/m°C, the density is 1050kg/m³, the heat capacity is 3.35kJ/kg°C, and the heat transfer coefficient at the surface of the hot dog is 3000W/m²°C.

Solution

Step 1
Draw the process diagram:
Step 2
Identify the geometry of the system.
The hot dog has a cylindrical shape with \( L/D \) equal to:

\[
\frac{L}{D} = \frac{0.16\text{ m}}{0.015\text{ m}} = 10.7
\]

Since \( L/D > 10 \), the hot dog can be treated as an infinite cylinder. The contribution of heat transferred through the bases of the cylinder can be neglected.

Step 3
Examine the surface temperature.
Since the surface temperature is unknown, the Biot number must be calculated:

\[
\text{Bi} = \frac{hR}{k} = \frac{3000\text{ W/m}^2\text{C}}{0.5\text{ W/m}^\circ\text{C}} = 45
\]

Since \( \text{Bi} > 40 \) the external resistance to heat transfer is negligible. Therefore, it can be assumed that the surface temperature will immediately reach the temperature of the environment, 100\(^\circ\text{C}\).

Step 4
Select the appropriate equation to use. The solution for constant surface temperature will be applied (Table 10.1):

\[
\frac{T_e - T}{T_e - T_0} = \frac{2}{R} \sum_{n=1}^{\infty} J_0(r\delta_n) \exp\left(-R^2\delta_n^2\text{Fo}\right)
\]  

(10.2)

Step 5
Calculate the Fourier number.
Calculate the thermal diffusivity first:

\[
\alpha = \frac{k}{\rho c_p} = \frac{0.5\text{ W/m}^\circ\text{C}}{1050\text{ kg/m}^3(3350\text{ J/kg}^\circ\text{C})} = 1.42 \times 10^{-7}\text{ m}^2/\text{s}
\]

Then:

\[
\text{Fo} = \frac{\alpha t}{R^2} = \frac{(1.42 \times 10^{-7}\text{ m}^2/\text{s})(120\text{ s})}{0.0075^2\text{ m}^2} = 0.30
\]

Since \( \text{Fo} > 0.2 \), only the first term of the series solution can be used without appreciable error. Alternatively, the solution can be read directly from a chart
such as Gurney-Lurie or Heisler (see Fig A.3 and A.6 in the Appendix), which
gives the dimensionless ratio \( \frac{T_e - T}{T_e - T_o} \) vs. the Fourier number.

1) Calculate the temperature 3 mm under the surface.

   i) From eqn (10.2) above:

Step 1
Find the values of \( \delta_n \).
Since \( Fo > 0.2 \) the first term of the series solution is enough. However,
two terms will be used in this problem for demonstration purposes. Since
\( x = R \delta_n \) and \( R = 0.0075 \), the values of \( \delta_n \) will be (see Table A.1 in the
Appendix):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \delta_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4048</td>
<td>320.6</td>
</tr>
<tr>
<td>5.5201</td>
<td>736.0</td>
</tr>
</tbody>
</table>

Step 2
Substitute these values into eqn (10.2) and calculate \( T \):

\[
\frac{T_e - T}{T_e - T_o} = \frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(r \delta_n)}{R \delta_n J_1(R \delta_n)} \exp\left(-R^2 \delta_n^2 Fo\right)
\]

\[
\approx \frac{2}{0.0075} \left( \frac{J_0(0.0045 \times 320.6)}{320.6 \times J_1(2.4048)} \exp(- (2.4048^2)(0.30)) + \frac{J_0(0.0045 \times 736)}{736 \times J_1(5.5201)} \exp(- (5.5201^2)(0.30)) \right)
\]

\[
= 266.67 \left( \frac{J_0(1.4427)}{320.6 \times J_1(2.4048)} \exp(-1.7349) + \frac{J_0(3.312)}{736 \times J_1(5.5201)} \exp(-9.141) \right)
\]

\[
= 266.67 \left( \frac{0.5436}{320.6 \times 0.5192} \exp(-1.7349) + \frac{-0.3469}{736 \times (-0.3403)} \exp(-9.141) \right)
\]

\[
= 266.67 (5.76 \times 10^{-4} + 1.48 \times 10^{-7}) = 0.154
\]

or

\[
\frac{T_e - T}{T_e - T_o} = \frac{100 - T}{100 - 5} = 0.154 \rightarrow T = 85.4^\circ C
\]
Comment: Notice that the 2nd term \((1.48 \times 10^{-7})\) is much smaller than the first term \((5.76 \times 10^{-4})\) and could have been neglected.

ii) From the Gurney-Lurie chart for an infinite cylinder (see Fig A.3 in the Appendix):

Step 1
Calculate the ratio \(r/R\) which gives the position of the point \((r\) is the distance of the point of interest from the axis of the cylinder):

\[
\frac{r}{R} = \frac{0.0045\,\text{m}}{0.0075\,\text{m}} = 0.6
\]

Step 2
Select the group of curves on the Gurney-Lurie chart that will be used based on the \(k/hR\) value.

Since \(\text{Bi} > 40\), the curves with \(k/hR = 0\) on the Gurney-Lurie chart can be used.

Step 3
On the Gurney-Lurie chart:
- Find the value of \(F_0 = 0.3\) on the x-axis
- Find the curve with \(k/hR = 0\) and \(r/R = 0.6\)
- Read the dimensionless temperature on the y-axis as \((T_e - T)/(T_e - T_o) = 0.15\)
- Calculate the temperature as

\[
\frac{T_e - T}{T_e - T_o} = \frac{100 - T}{100 - 5} = 0.15 \rightarrow T = 85.8^\circ\text{C}
\]

2) Calculate the temperature at the center.

i) From the equation:
As above but with \(r = 0\) :
\[
\frac{T_e - T}{T_e - T_0} = \frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(r_0^a)^n}{\delta_n J_1(R_0^a)} \exp\left(-R^2 \delta_n^2 \text{Fo}\right)
\]

\[
\approx \frac{2}{0.0075} \left( \frac{J_0(0 \times 320.6)}{320.6 \times J_1(2.4048)} \exp\left(-\left(2.4048^2\right) \times 0.3\right)\right)
\]

\[+ \frac{J_0(0 \times 736)}{736 \times J_1(5.5201)} \exp\left(-\left(5.5201^2\right) \times 0.3\right)\right)\]

\[= 266.67 \left( \frac{J_0(0)}{320.6 \times J_1(2.4048)} \exp\left(-1.7349\right) + \frac{J_0(0)}{736 \times J_1(5.5201)} \exp\left(-9.141\right)\right)\]

\[= 266.67 \left( \frac{1}{320.6 \times 0.5192} \exp\left(-1.7349\right) + \frac{1}{736 \times (-0.3403)} \exp\left(-9.141\right)\right)\]

\[= 266.67 \left(1.06 \times 10^{-3} - 4.279 \times 10^{-7}\right) = 0.283\]

or

\[
\frac{T_e - T}{T_e - T_0} = \frac{100 - T}{100 - 5} = 0.283 \rightarrow T = 73.1^\circ\text{C}
\]

**Comment:** Notice that the 2nd term \((4.279 \times 10^{-7})\) is much smaller than the first term \((1.06 \times 10^{-3})\) and could have been neglected.

ii) From the Heisler chart for an infinite cylinder (Fig A.6):

On the Heisler chart:

- Find the value of \(\text{Fo} = 0.3\) in the x-axis.
- Find the curve with \(k/hR = 0\).
- Read the dimensionless temperature on the y-axis as \((T_e - T)/(T_e - T_0) = 0.27\)
- Calculate the temperature as

\[
\frac{T_e - T}{T_e - T_0} = \frac{100 - T}{100 - 5} = 0.27 \rightarrow T = 74.4^\circ\text{C}
\]
3) Calculate the time necessary to reach 81°C at the center:

Calculate the dimensionless temperature

\[
\frac{T_c - T}{T_c - T_o} = \frac{100 - 81}{100 - 5} = 0.2
\]

On the Heisler chart:

- Find on the y-axis the value \( (T_c - T)/(T_c - T_o) = 0.2 \)
- Find the curve \( k/hR = 0 \).
- Read the Fourier number on the x-axis as \( Fo = 0.36 \).
- Find the time from the Fourier number as:

\[
t = Fo \frac{R^2}{\alpha} = 0.36 \frac{0.0075^2 \, m^2}{1.42 \times 10^{-7} \, m^2/s} = 143 \, s
\]

4) Calculate the average temperature.

Step 1

Write the equation for the mean temperature of an infinite cylinder with \( Bi > 40 \) (Table 10.1):

\[
\frac{T_c - T_m}{T_c - T_o} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\pi^2}{R^2 \delta_n^2} \exp(-R^2 \delta_n^2 Fo)
\]

\[
= \frac{4}{\pi^2} \left( 1.7066 \exp(-5.783Fo) + 0.324 \exp(-30.5Fo) + 0.132 \exp(-74.9Fo) + ... \right)
\]

Step 2

Substitute values in the above equation and solve for \( T_m \) (since \( Fo > 0.2 \), the first term only is enough; in this example, three terms will be used for demonstration purposes):
\[
\frac{T_e - T_m}{T_c - T_o} = \frac{4}{\pi^2} \left( 1.7066 \exp(-5.783 \times 0.3) + 0.324 \exp(-30.5 \times 0.3) \\
+ 0.132 \exp(-74.9 \times 0.3) + ... \right)
\]
\[
= \frac{4}{\pi^2} \left( 0.3011 + 3.44 \times 10^{-5} + 2.3 \times 10^{-11} + ... \right) = 0.122
\]

and

\[
T_m = T_e - 0.122(T_e - T_o) = 100 - 0.122(100 - 5) = 88.4^\circ C
\]

**Comment:** Notice that the 2\textsuperscript{nd} and 3\textsuperscript{rd} terms of the sum are negligible compared to the 1\textsuperscript{st} term and they could have been omitted.

5) Calculate the heat transferred to the solid in 2 min.

Because

\[
\frac{q_t}{q_e} = \frac{mc_p(T_m - T_o)}{mc_p(T_e - T_o)} = \frac{T_m - T_o}{T_e - T_o} = 1 - \frac{T_c - T_m}{T_e - T_o}
\]

the heat transferred to the hot dog in 2 min will be:

\[
q_t = mc_p(T_e - T_o) \left( 1 - \frac{T_c - T_m}{T_e - T_o} \right)
\]
\[
= (V \rho) c_p (T_e - T_o) \left( 1 - \frac{T_c - T_m}{T_e - T_o} \right)
\]
\[
= \left( \frac{\pi (0.015^2)}{4} \right) (0.16 \text{ m}) \left( 1050 \frac{\text{kg}}{\text{m}^3} \right) \left( 3350 \frac{\text{J}}{\text{kg} \cdot \text{C}} \right) (100 - 5 ^\circ \text{C})(1 - 0.122)
\]
\[
= 8296 \text{ J}
\]

**Example 10.3**

12mm × 16mm × 14mm rectangular fruit pieces are immersed in syrup. Calculate the temperature at the center of a piece of fruit after 5 min if the initial temperature of the fruit piece is 20\(^\circ\)C, the syrup temperature is 100\(^\circ\)C, the heat transfer coefficient at the surface of the fruit piece is 83\text{W}/\text{m}^2\cdot\text{C}, and the physical properties of the fruit piece are \(k = 0.5 \text{ W/m} \cdot \text{C}, \ c_p = 3.8 \text{ kJ/kg} \cdot \text{C}, \) and \(\rho = 900 \text{ kg/m}^3. \) Assume the fruit piece does not exchange matter with the syrup.

**Solution**

Step 1

Draw the process diagram:
The temperature at the center is affected by the heat transferred from the three directions x, y, and z. The contribution from each direction has to be calculated separately, and the combined effect will be calculated at the end.

1) Heat transferred in the x-direction:

Step 1
Select the appropriate equation to use.

i) Calculate the Biot number for the x-direction:

\[ Bi_x = \frac{hL_x}{k} = \left( \frac{83 \text{ W/m}^2 \text{°C}}{0.5 \text{ W/m°C}} \right) \left( 0.006 \text{ m} \right) = 1 \]

ii) Since 0.1 < Bi < 40 both external and internal resistances are important. Therefore, the solution will be (Table 10.2):

\[ \frac{T_e - T_x}{T_e - T_o} = \sum_{n=1}^{\infty} \frac{2Bi_x}{\delta_n^2 + Bi_x + Bi_x} \frac{\cos \left( \frac{X}{L_x} \delta_n \right)}{\cos (\delta_n)} \exp \left( -\frac{\delta_n^2}{\alpha} \right) \]

Step 2
Calculate the Fourier number \( F_{ox} \) for the x-direction:

i) Calculate the thermal diffusivity:

\[ \alpha = \frac{k}{\rho c_p} = \frac{0.5 \text{ W/m°C}}{\left( \frac{900 \text{ kg/m}^3 \right) \left( 3800 \text{ J/kg°C} \right) = 1.46 \times 10^{-7} \text{ m}^2/\text{s} \]
ii) Calculate the Fourier number:

$$\text{Fo}_x = \frac{\alpha t}{L_x^2} = \frac{(1.46 \times 10^{-7} \text{ m}^2/\text{s})(300 \text{ s})}{0.006^2 \text{ m}^2} = 1.217$$

Since $\text{Fo}_x > 0.2$, only the first term of the sum in the above equation can be used without appreciable error. Alternatively, the solution can be read directly from the Heisler chart.

Step 3
Calculate the temperature:

i) From the equation:

The first root of the equation $\delta \tan \delta = \text{Bi}$ for $\text{Bi} = 1$ is: $\delta_1 = 0.8603$ (see Table A.2 in the Appendix).

$$\frac{T_e - T_x}{T_e - T_o} \approx \frac{2 \text{Bi}_x}{\delta^2_n + \text{Bi}^2_x + \text{Bi}_x} \cos\left(\frac{x \delta_n}{L_x}\right) \exp\left(-\delta^2_n \text{Fo}_x\right) =$$

$$= \frac{2 \times 1}{0.8603^2 + 1^2 + 1} \cos\left(\frac{0.006 \times 0.8603}{\cos(0.8603)}\right) \exp\left(-0.8603^2 \times 1.217\right)$$

$$= 0.455$$

ii) From the Heisler chart for an infinite slab:

- Find the value of $\text{Fo} = 1.22$ on the x-axis.
- Find the curve with $k/hL = 1$.
- Read the dimensionless temperature on the y-axis as:

$$\frac{T_e - T_x}{T_e - T_o} = 0.46$$
2) Heat transferred in the y-direction:

Step 1
Select the appropriate equation to use.

i) Calculate the Biot number for the y-direction:

\[ \text{Bi}_y = \frac{hL_y}{k} = \frac{\left(83 \text{ W/m}^2 \text{ °C}\right) (0.008 \text{ m})}{0.5 \text{ W/m}^\circ \text{C}} = 1.328 \]

ii) Since 0.1 < Bi < 40 both external and internal resistances are important. Therefore, the solution will be:

\[ \frac{T_e - T_y}{T_e - T_o} = \sum_{n=1}^{\infty} \frac{2 \times \text{Bi}_y}{\delta_n^2 + \text{Bi}_y + \text{Bi}_y} \frac{\cos \left( \frac{y L_y}{\delta_n} \right)}{\cos (\delta_n)} \exp \left( -\delta_n^2 \text{Fo}_y \right) \]

Step 2
Calculate the Fourier number Fo_y for the y-direction:

\[ \text{Fo}_y = \frac{\alpha t}{L_y^2} = \frac{(1.46 \times 10^{-7} \text{ m}^2/\text{s})(300 \text{ s})}{0.008^2 \text{ m}^2} = 0.684 \]

Since Fo_y > 0.2, only the first term of the sum in the above equation can be used without appreciable error. Alternatively, the solution can be read directly from the Heisler chart.

Step 3
Calculate the temperature:

i) From the equation:

The first root of the equation \( \delta \tan \delta = \text{Bi} \) for Bi = 1.328 is: \( \delta_1 = 0.9447 \) (see Table A.2).

\[ \frac{T_e - T_y}{T_e - T_o} \approx \frac{2 \times \text{Bi}_y}{\delta_1^2 + \text{Bi}_y + \text{Bi}_y} \frac{\cos \left( \frac{y L_y}{\delta_1} \right)}{\cos (\delta_1)} \exp \left( -\delta_1^2 \text{Fo}_y \right) \]

\[ = \frac{2 \times 1.328}{0.9447^2 + 1.328^2 + 1.328} \frac{\cos \left( \frac{0}{0.008} \times 0.9447 \right)}{\cos (0.9447)} \exp (-0.9447^2 \times 0.684) = 0.618 \]
ii) From the Heisler chart for an infinite slab (Fig. A.5):

- Find the value of $\text{Fo} = 0.68$ on the x-axis.
- Find the curve with $k/hL = 0.75$ (interpolate between curves 0.7 and 0.8).
- Read the dimensionless temperature on the y-axis as:

$$\frac{T_e - T_y}{T_e - T_o} = 0.61$$

3) Heat transferred in the z-direction:

Step 1
Select the appropriate equation to use.

i) Calculate the Biot number for the z-direction:

$$\text{Bi}_z = \frac{hL_z}{k} = \frac{(83 \text{ W/m}^2\text{C})(0.007 \text{ m})}{0.5 \text{ W/m}^\circ\text{C}} = 1.162$$

ii) Since $0.1 < \text{Bi} < 40$ both external and internal resistances are important. Therefore, the solution will be (Table 10.2):

$$\frac{T_e - T_z}{T_e - T_o} = \sum_{n=1}^{\infty} \frac{2\text{Bi}_z}{\delta_n^2 + \text{Bi}_z + \text{Bi}_z} \frac{\cos\left(\frac{z}{L_z} \delta_n\right)}{\cos(\delta_n)} \exp(-\delta_n^2 \text{Fo}_z)$$

Step 2
Calculate the Fourier number $\text{Fo}_z$ for the z-direction:

$$\text{Fo}_z = \frac{\alpha t}{L_z^2} = \frac{(1.46 \times 10^{-7} \text{ m}^2/\text{s})(300 \text{ s})}{0.007^2 \text{ m}^2} = 0.894$$
Since $F_o > 0.2$, only the first term of the sum in the above equation can be used without appreciable error. Alternatively, the solution can be read directly from the Heisler chart.

Step 3
Calculate the temperature:

i) From the equation:
The first root of the equation $\delta \tan \delta = Bi$ for $Bi = 1.162$ is: $\delta_1 = 0.9017$ (see Table A.2):

$$\frac{T_e - T_z}{T_e - T_o} \approx \frac{2Bi_z \cos \left( \frac{z}{L} \delta_n \right)}{\delta_n^2 + Bi_z^2 + Bi_z \cos(\delta_n)} \exp\left(-\delta_n^2 F_o\right) =$$

$$= \frac{2 \times 1.162}{0.9017^2 + 1.162^2 + 1.162} \cos\left(\frac{0}{0.007} \times 0.9017\right) \exp(-0.9017^2 \times 0.894) = 0.545$$

ii) From the Heisler chart for an infinite slab (Fig. A.5):

- Find the value of $F_o = 0.89$ on the x-axis.
- Find the curve with $k/hL = 0.86$ (interpolate between curves 0.8 and 1.0).
- Read the dimensionless temperature on the y-axis as:

$$\frac{T_e - T_z}{T_e - T_o} = 0.55$$
4) The combined effect of heat transferred in the x, y, and z directions is:

\[
\frac{T_c - T_{xyz}}{T_c - T_o} = \left(\frac{T_c - T_x}{T_c - T_o}\right) \left(\frac{T_c - T_y}{T_c - T_o}\right) \left(\frac{T_c - T_z}{T_c - T_o}\right)
\]

\[
= 0.455 \times 0.618 \times 0.545 = 0.153
\]

and the temperature is:

\[
\frac{100 - T_{xyz}}{100 - 20} = 0.153 \rightarrow T_{xyz} = 87.8 \, ^\circ C
\]

**Example 10.4**

Calculate how long it will take for the temperature on the non-heated surface of the steak of Example 10.1 to increase by 1 % of the initial temperature difference.

*Solution*

Step 1

Identify the geometry of the system.

As long as the temperature change on the cold surface of the steak is less than 1\% of the initial temperature difference, the steak can be treated as a semi-infinite body.

Step 2

Examine the surface temperature.

The surface temperature is constant at 120\(^\circ\)C.

Step 3

Select the appropriate equation.

The equation for a semi-infinite body with uniform initial temperature and constant surface temperature is (Table 10.4):

\[
\frac{T_c - T}{T_c - T_o} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)
\]

with

\[
\frac{T - T_o}{T_c - T_o} = 0.01
\]

(since the accomplished temperature change is 1\% )
or

\[
\frac{T_e - T}{T_e - T_o} = 1 - \frac{T - T_o}{T_e - T_o} = 1 - 0.01 = 0.99
\]

Therefore

\[
0.99 = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)
\]

Step 4
Find the erf of the argument \(x/2\sqrt{\alpha t}\) from an erf table (Table A.5).
The argument has to be equal to 1.82 for the error function to be equal to 0.99.
Therefore

\[
\frac{x}{2\sqrt{\alpha t}} = 1.82
\]

Step 5
Solve for \(t\):

\[
t = \frac{(\frac{x}{3.64})^2}{\alpha} = \frac{(0.02 \text{ m})^2}{1.4 \times 10^{-7} \text{ m}^2/\text{s}} = 215.6 \text{ s}
\]

Comment: The distance \(x\) where the temperature has increased by 1% of the initial temperature difference is called the “thermal penetration depth.” As long as the thermal penetration depth in a finite body is less than the thickness of the body (or half the thickness of the body, in the case of heating from both sides), the body can be treated as a semi-infinite body.

Example 10.5

Concentrated milk is sterilized in a can with 7.5 cm diameter and 9.5 cm height. Calculate the time required to heat the milk from 45°C to 115°C given that: a) the can is in such a motion during heating that the milk temperature is uniform inside the can, b) the can contains 410 g of milk, c) the overall heat transfer coefficient between the heating medium and the milk is 300 W/m²°C, d) the heating medium temperature is 130°C, and e) the mean heat capacity of the milk over the temperature range of the heating process is 3650 J/kg°C.
Solution

Since the milk temperature is uniform inside the can during heating, the internal resistance can be considered negligible. The solution for $\text{Bi} < 0.1$ can be applied (Table 10.3). Thus,

$$t = \frac{mc_p}{hA} \ln \frac{T_e - T_o}{T_e - T} =$$

$$= \frac{(0.410 \text{ kg}) \left(3650 \text{ J/kg}^\circ\text{C}\right)}{\left(300 \text{ W/m}^2\circ\text{C}\right) \left(\pi (0.075 \text{ m})(0.095 \text{ m}) + 2 \left(\pi \frac{0.075^2}{4}\right) \text{ m}^2\right)} \ln \frac{130 - 45}{130 - 115}$$

$$= 277 \text{ s}$$

Exercises

Exercise 10.1

Peas are blanched by immersion in hot water at 90\(^\circ\)C. Calculate the temperature at the center of a pea after 3 min if the diameter of the pea is 8 mm, the initial temperature of the pea is 20\(^\circ\)C, and the heat transfer coefficient at the surface of the pea is 100W/m\(^2\)C. Assume that the physical properties of the pea are $\rho = 1050\text{ kg/m}^3$, $c_p = 3.7\text{kJ/kg}^\circ\text{C}$, $k = 0.5\text{ W/m}^\circ\text{C}$.

Solution

Step 1
State your assumptions.
The pea can be assumed to be a sphere.

Step 2
Select the equation for a sphere.

i) Calculate the Biot number:

$$\text{Bi} = \text{.........................}$$

Since $0.1 < \text{Bi} < 40$, internal and external resistances are important.

ii) The equation to be used is:

$$\frac{T_e - T}{T_e - T_o} = \sum_{n=1}^{\infty} \frac{2 \text{Bi}}{\delta_n^2 + \text{Bi}^2 - \text{Bi}} \frac{\delta_n}{\sin(\delta_n)} \exp(-\delta_n^2 \text{Fo})$$
Step 3
Calculate the Fourier number:

\[ \alpha = \frac{C_11}{C_0} = \ldots \]

and

\[ \text{For} = \ldots \]

Since \( \text{Fo} > 0.2 \), only the first term of the sum in the above equation can be used without appreciable error. Alternatively, the solution can be read directly from the Heisler chart.

Step 4
Calculate the temperature:

i) From the above equation:
   Substitute values in the above equation \((\delta_1 = 1.432)\) and calculate the temperature as

\[ \frac{90 - T}{90 - 20} \approx \ldots \]

\[ T = \ldots \] \(^\circ\text{C}\)

ii) Also find the temperature from the Heisler chart for a sphere:
   - Find the value of Fo on the x-axis.
   - Calculate \(k/hR\) and find the corresponding curve.

\[ \frac{k}{hR} = \ldots \]

- Read the dimensionless temperature on the y-axis as:

\[ \frac{T_e - T}{T_e - T_o} = \ldots \]

- Calculate the temperature as:

\[ \ldots - T = \ldots \]

\[ \rightarrow T = \ldots \]
Exercise 10.2

A 211x300 can containing a meat product is heated in a retort. The initial temperature of the can is 50°C, the heat transfer coefficient at the surface is 3000 W/m²°C, and the physical properties of the meat product are: $k = 0.5$ W/m°C, $c_p = 2.85$ kJ/m°C, and $\rho = 1100$ kg/m³. Calculate the temperature at the geometric center of the can after 30 min if the steam temperature in the retort is 130°C.

Solution

Step 1
Determine the shape of the object.
The shape of the can is a finite cylinder. The temperature at the center is affected by the heat transferred from the cylindrical surface as well as from the flat bases of the cylinder. The contribution from each direction has to be calculated separately and the combined effect will be calculated at the end.

The dimensions of the can are:

- Diameter = $2 \frac{11}{16}$ inches = 0.06826 m,
- Height = $3 \frac{0}{16}$ inches = 0.0762 m

1) Calculate the heat transferred in the radial direction.

Step 1
Select the appropriate equation:

i) Calculate the Biot number for the r-direction and determine the relative significance of external and internal resistances:

$$\text{Bi}_r = \ldots$$
ii) Select the equation:

Step 2
Calculate the Fourier number $F_{or}$ for the radial direction:

$$F_{or} = \ldots$$

Is $F_{or} > 0.2$? If yes, only the first term of the sum in the above equation can be used without appreciable error. Alternatively, the solution can be read directly from the Heisler chart.

Step 3
Calculate the temperature:

i) From the equation:

Find the values of $\delta n$. As in Example 10.2, substitute values in the selected equation and calculate:

$$\frac{T_e - T_r}{T_c - T_o} = \ldots$$

ii) From the Heisler chart:

On the x-axis of the Heisler chart for a infinite cylinder:

- Find the value of $F_{or}$.
- Calculate $k/hR$ and find the corresponding curve.
- Read the dimensionless temperature on the y-axis as:

$$\frac{T_e - T_r}{T_c - T_o} = \ldots$$
2) Calculate the heat transferred in the x-direction.

Step 1
Select the appropriate equation:

i) Calculate the Biot number for the x-direction and determine the relative significance of external and internal resistances:

\[ \text{Bi}_x = \ldots \]

ii) Select the equation:

\[ \ldots \]

Step 2
Calculate the Fourier number \( \text{Fo}_x \) for the x-direction:

\[ \text{Fo}_x = \ldots \]

Is \( \text{Fo}_x > 0.2 \)? If yes, only the first term of the sum in the above equation can be used without appreciable error. Alternatively, the solution can be read directly from the Heisler chart.

Step 3
Calculate the temperature:

i) From the equation:

Substitute values and calculate:

\[ \frac{T_e - T_x}{T_e - T_0} = \ldots \]

ii) From the Heisler chart:

On the x-axis of the Heisler chart for an infinite slab:

- Find the value of \( \text{Fo}_x \).
- Calculate \( k/hL_x \) and find the corresponding curve.
- Read the dimensionless temperature on the y-axis as:

\[ \frac{T_e - T_x}{T_e - T_0} = \ldots \]
3) Calculate the combined effect of the heat transferred through the cylindrical surface and the heat transferred through the flat bases of the can:

\[
\frac{T_e - T_{rx}}{T_e - T_o} = \left( \frac{T_e - T_r}{T_e - T_o} \right) \left( \frac{T_e - T_x}{T_e - T_o} \right) = \ldots \ldots \ldots \ldots \ldots \ldots
\]

Calculate the temperature as:

\[
\frac{130 - T_{rx}}{\ldots} = \ldots \ldots \ldots \ldots \ldots \ldots \rightarrow T_{rx} = \ldots \ldots \ldots \ldots ^\circ C
\]

Exercise 10.3

For the removal of field heat from fruits, hydrocooling is usually used. Calculate the time required to reduce the center temperature of a spherical fruit from 25°C to 5°C by immersion in cold water of 1°C if the diameter of the fruit is 6 cm, the heat transfer coefficient is 1000 W/m²°C, and the physical properties of the fruit are: \( k = 0.4 \text{W/m}^\circ \text{C} \), \( \rho = 900 \text{kg/m}^3 \), \( c_p = 3.5 \text{kJ/kg}^\circ \text{C} \).

Solution

Solve the problem using the Heisler chart for a sphere:

- Calculate the value of the dimensionless temperature:

\[
\frac{T_e - T}{T_e - T_o} = \ldots \ldots \ldots \ldots \ldots \ldots
\]

- Find the value of the dimensionless temperature on the y-axis of the Heisler chart:
- Calculate the value of \( k/hR \) and find the corresponding curve.

\[
\frac{k}{hR} = \ldots \ldots \ldots \ldots \ldots \ldots
\]
- Read the Fourier number on the x-axis (check if Fo > 0.2 so that the Heisler chart is valid).
- Calculate the time from:

\[ t = \frac{\text{Fo} R^2}{\alpha} = \ldots \]

**Exercise 10.4**

A root crop is in the ground 5 cm from the surface when suddenly the air temperature drops to \(-10^\circ\text{C}\). If the freezing point of the crop is \(-1^\circ\text{C}\), calculate if the temperature of the soil at this depth will drop below the freezing point after 24 h. The initial temperature of the soil is \(12^\circ\text{C}\), the heat transfer coefficient at the surface of the ground is \(10\,\text{W/m}^2\,^\circ\text{C}\), and the physical properties of the soil are \(k = 0.5\,\text{W/m}^\circ\text{C}\), \(\rho = 2000\,\text{kg/m}^3\) and \(c_p = 1850\,\text{J/kg}^\circ\text{C}\). Neglect latent heat effects.

**Solution**

Step 1

Draw the process diagram:
Step 2
Select the appropriate equation to use. The ground can be treated as a semi-infinite body. Therefore, the equation for unsteady state in a semi-infinite body with convection at the surface (since the surface temperature is not constant) will be used (Table 10.4):

\[ \frac{T_e - T}{T_e - T_o} = \text{erf} \left( \frac{x}{2\sqrt{\alpha t}} \right) + \left[ \exp \left( \frac{h x}{k} + \frac{h^2 \alpha t}{k^2} \right) \right] \left[ \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \right] \]  

(10.3)

Step 3
Calculate the thermal diffusivity:

\[ \alpha = \ldots = \ldots \text{ m}^2/\text{s} \]

Step 4
Calculate:

\[ \frac{x}{2\sqrt{\alpha t}} = \frac{0.05 \text{ m}}{\ldots} = \ldots \]

and

\[ \frac{h\sqrt{\alpha t}}{k} = \frac{(10 \text{ W/m}^2 \circ \text{oC}) \times \ldots}{\ldots} = \ldots \]

Step 5
Find the value of \( \text{erf} \left( \frac{x}{2\sqrt{\alpha t}} \right) \) and \( \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \) from Table A.5 taking into account that \( \text{erfc} = 1 - \text{erf} \).

Step 6
Substitute values in eqn (10.3) above and calculate:

\[ \frac{T_e - T}{T_e - T_o} = \ldots \]

\[ \rightarrow T = \ldots \circ \text{C} \]
Exercise 10.5

Calculate the temperature of 200 litres of glucose syrup heated in an agitated jacketed vessel for 30 min if the initial temperature of the syrup is 30°C, the temperature of the steam in the jacket is 120°C, the density and the heat capacity of the syrup are 1230kg/m³ and 3.35kJ/kg°C respectively, the overall heat transfer coefficient between the heating medium and the syrup is 200W/m²°C, and the inside surface area of the vessel in contact with the syrup is 1.5 m².

Solution

Step 1
Select the appropriate equation to use. Since the syrup is agitated, its temperature can be assumed to be uniform. Therefore the equation to use is:

Step 2
Calculate the mass of the syrup:

\[ m = V \rho = \text{.................................} \]

Step 3
Apply the selected equation:

\[ \frac{T_c - T}{T_e - T_0} = \text{.................................} \]

\[ \rightarrow T = \text{.........................}^\circ \text{C} \]
Exercise 10.6

A thermocouple with 4 mm diameter and 2 cm length is immersed in milk which is at 60°C. How long will it take for the thermocouple to reach a temperature of 59.5°C if the heat transfer coefficient at the surface of the thermocouple is 200W/m²°C, the initial temperature of the thermocouple is 25°C, its mass is 1 g, its heat capacity is 0.461kJ/kg°C, and its thermal conductivity is 15W/m°C. If the thermocouple were exposed to air of 60°C instead of milk, how long would it take to reach 59.5°C? Assume that the heat transfer coefficient in this case is 20W/m²°C.

Solution

Step 1
Select the appropriate equation:

i) Calculate the Biot number:

\[ Bi = \frac{hD}{\lambda} \]

ii) Determine the relative importance of internal and external resistance:

\[ \frac{D}{\lambda} \]

iii) Select and apply the appropriate equation:

\[ \frac{D}{\lambda} \]

Step 2
For the case of air, repeat the calculations with \( h = 20 \text{W/m}^2\text{°C} \).

Exercise 10.7

Solve Example 10.1 using the spreadsheet program Heat Transfer-Negligible Surface Resistance.xls. Also solve the problem for the case in which the thickness of the steak is 3 cm and for the case in which the steak is heated from both sides.

Solution
Step 1
"Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).

Step 2
Insert the parameter values in the yellow cells using: half thickness in the x-direction $L_x = 0.02$ m (since the steak is heated from one side, $L_x$ is equal to the thickness); half thickness in the y-direction and z-direction $L_y = 1$ m, $L_z = 1$ m (since the steak is considered as an infinite plate, use big numbers on $L_y$ and $L_z$ to simulate the fact that heat transfer is significant only in the x-direction); distance of the point from the coldest plane in the x-direction $x = 0.02 - 0.011 = 0.009$ m; distance in the z and y directions $y = 0$ and $z = 0$ (since the steak is considered as an infinite plate); initial temperature $T_o = 5^\circ$C; surface temperature $T_e = 120^\circ$C. Also insert the given values for the thermal diffusivity.

Step 3
Set the value of the time step to $15 \times 60 = 900$ s in cell F4.

Step 4
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 5
Read the value for the temperature of the point that lies 9 mm below the surface in cell K20 (green cell).

Step 6
Read the value for the mean temperature in cell K40 (green cell).

Step 7
To see the effect of steak thickness, set $L_x = 0.03$. Repeat steps 2 to 6.

Step 8
To see how the temperature of the point changes with time:

a) Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).
b) Set the value of the time step in cell F4 equal to 10 or some other small value.
c) Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).
d) Iterate by pressing F9 until the temperature in cell K20 reaches the value you want.
Step 9
To see how much faster the temperature at the same point would have increased if the steak were heated from both sides, i.e., between the hot plates of a toaster:

\[
L_x \quad x
\]

a) Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).
b) Set the value of half thickness in the x-direction to \( L_x = 0.01 \) m.
c) Set the value of the coordinates of the point to \( x = 0.001 \) m, \( y = 0 \), \( z = 0 \).
d) Set the value of the time step equal to 10.
e) Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).
f) Iterate by pressing F9 until the same temperatures as in steps 5 and 6 are reached.
g) Read the time in cell F3.

Exercise 10.8

Solve Example 10.2 using the spreadsheet program Heat Transfer-Negligible Surface Resistance.xls.

Solution

Step 1
On the spreadsheet Heat Transfer-Negligible Surface Resistance.xls, go to the sheet “cylinder.” Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).

Step 2
Insert the parameter values in the yellow cells using: diameter of the cylinder \( D = 0.015 \) m; half of the height of the cylinder \( L_x = 0.08 \) m; distance of the point from the center line in the direction of radius \( r = 0.0045 \) m.
(point 3 mm from the surface); distance of the point from the center of the cylinder in the x direction $x = 0$; initial temperature $T_o = 5^\circ C$; surface temperature $T_e = 100^\circ C$. Also insert the given values for the physical properties and the heat transfer coefficient.

Step 3
Read the Biot number for the r and x directions in cells B42 and C42

Step 4
Set the value of the time step to 120 in cell F4.

Step 5
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 6
Read the value for the temperature of the point that lies 3 mm below the surface in cell K20 (green cell).

Step 7
Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).

Step 8
Set the value of $r = 0$ and $x = 0$ (coordinates of the center of the cylinder).

Step 9
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 10
Read the value for the temperature of the center of the hot dog in cell K20 (green cell).

Step 11
Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).

Step 12
Set the value of the time step to 0.5 in cell F4.

Step 13
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 14
Iterate by pressing F9 until the temperature in cell K20 is equal to $81^\circ C$. Read the value of the time in cell F3.

Step 15
Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).
Step 16
Set the value of the time step to 120 in cell F4.

Step 17
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 18
Read the value for the mean temperature in cell K40 (green cell).

Step 19
Read the value for the heat transferred in cell K43.

Exercise 10.9

Solve Example 10.3 using the spreadsheet Heat Transfer-Internal and External Resistance.xls. Run the program for $h = 50 \text{W/m}^2 \text{C}$ and $h = 200 \text{W/m}^2 \text{C}$ to see the effect of the heat transfer coefficient on the time-temperature relationship.

Solution

Step 1
On the spreadsheet Heat Transfer-Internal and External Resistance.xls go to the sheet “slab”.
Turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).

Step 2
Insert the parameter values in the yellow cells using: $L_x = 0.006 \text{ m}; L_y = 0.008 \text{ m}; L_z = 0.007 \text{ m};$ distance of the point from the geometric center $x = 0, y = 0, z = 0$; initial temperature $T_o = 20^\circ \text{C}$; temperature of the environment $T_e = 100^\circ \text{C}$. 
Also insert the given values for the density, the heat capacity, the thermal conductivity, and the heat transfer coefficient.

Step 3
Read the Biot numbers for the x, y, and z direction in cells B37, C37 and D37. Are they between 0.1 and 40?

Step 4
Set the value of the time step to \( 5 \times 60 = 300 \) in cell F4.

Step 5
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 6
Read the temperature for the center in cell K15 (green cell).

Step 7
Read the mass average temperature in cell K30 (green cell).

Step 8
To see how the temperature at the selected point changes with time turn the “SWITCH” OFF (set the value in cell G1 equal to 0 and press ENTER).

Step 9
Set the value of the time step equal to 10 (or some other small value) in cell F4.

Step 10
Turn the “SWITCH” ON (set the value in cell G1 equal to 1 and press ENTER).

Step 11
Iterate (by pressing F9) until the value of time in cell F3 (grey cell) reaches 300.

Step 12
See the plot for the center temperature and mass average temperature vs. time and the dimensionless temperature in the x-direction vs. Fourier number for the x-direction in the diagrams.

Step 13
Change the value of \( h \) and the corresponding values of \( \delta i \). Run the program and see the temperature at the center after 5 min.

Step 14
Make the necessary changes in the spreadsheet to plot the dimensionless non-accomplished temperature change for the y and z directions.

**Exercise 10.10**

For the removal of field heat from certain fruits, hydrocooling, forced air cooling and room cooling have been proposed. Calculate and plot the temperature change at the center of the fruit vs. the time for the three methods. Assume
that the fruit is spherical; its initial temperature is 25°C; the cold water and the air temperature are 1°C; the diameter of the fruit is 6 cm; the heat transfer coefficient is 1000 W/m²°C in the case of hydrocooling, 20 W/m²°C in the case of forced air cooling, and 5 W/m²°C in the case of room cooling; and the physical properties of the fruit are: \( k = 0.4 \text{ W/m°C} \), \( \rho = 900 \text{ kg/m}^3 \), \( c_p = 3.5 \text{ kJ/kg°C} \).

Solution

![Diagram of fruit with radius and diameter labeled]

Step 1
Calculate the Biot number in all cases.

Step 2
Select the spreadsheet \textit{Heat Transfer-Negligible Surface Resistance.xls} or \textit{Heat Transfer-Internal and External Resistance.xls} depending on the value of Biot number.

Step 3
Follow the instructions and run the program for a sphere.

Step 4
Run the program using a small time increment, e.g., 10 s.

Exercise 10.11

The product development department in a food plant is interested in knowing what shape they should give to a new product for a faster temperature change at the center of the product from 20°C to 90°C. Should the shape be a cube 50 mm × 50 mm × 50 mm, a cylinder with diameter 62.035 mm and length 41.36 mm, or a sphere with diameter 62.035 mm? All these shapes will have
the same volume and the same mass. The heating medium is steam at 100°C; the heat transfer coefficient in all cases is 2000W/m²°C; and the density, the heat capacity, and the thermal conductivity of the product are 900kg/m³, 3200J/kg°C, and 0.5W/m°C respectively. Determine which shape will give a faster temperature change at the center.

**Solution**

Step 1
Calculate the Biot number in all cases.

Step 2
Select the spreadsheet *Heat Transfer-Negligible Surface Resistance.xls* or *Heat Transfer-Internal and External Resistance.xls* depending on the value of the Biot number.

Step 3
Follow the instructions and run the program for a cube, a cylinder, and a sphere.

Step 4
Find the time necessary in each case for $T_{local}$ to reach 90°C.

**Exercise 10.12**

Understand how the spreadsheet program *Heat Transfer-Negligible Surface Resistance.xls* calculates and plots the temperature change of a sphere with time. Modify the program to plot the temperature distribution with the radius of the sphere for a certain time.

**Exercise 10.13**

Understand how the spreadsheet program *Heat Transfer-Internal and External Resistance.xls* calculates and plots the temperature change of a cylinder with time. Modify the program to plot $q_t/q_e$ vs. $Fo^{0.5}$ for a certain Biot number value for an infinite cylinder.

**Exercise 10.14**

a) Solve Exercise 10.4 using the spreadsheet program *Semi-Infinite body 1.xls*.
b) Plot the temperature profile for the depth 0-5 cm after 12 h and 24 h. c) What is the minimum depth at which the temperature does not change after 12 h?
d) Observe how the temperature changes with time at the depth of 1 cm, 3 cm, and 5 cm for the first 24 hours.
Solution

Step 1
In the sheet “T vs. x”, follow the instructions, run the program for Time = 24 \times 3600 = 86400 s and read the temperature in cell F16 when the distance in cell F3 is equal to 0.05 m. Observe the temperature profile in the plot.

Step 2
Set the time equal to 12 \times 3600 = 43200 s in cell B29, follow the instructions, run the program, and observe the temperature profile in the plot.

Step 3
Continue running the program until the temperature in cell F16 is equal to 12\degree C. Read the depth in cell F3.

Step 4
In the sheet “T vs. t”, follow the instructions, run the program for Distance x = 0.01 m until Time = 86400 s in cell F3. Observe the temperature profile in the plot. Rerun the program for x = 0.03 m and x = 0.05 m following the instructions.

Exercise 10.15
Use the spreadsheet program *Semi-Infinite body 2.xls*, to study how the temperature in a semi-infinite body changes with time when a) the temperature at the surface is constant, and b) the temperature at the surface is not constant but is affected by convection. Find the value of \( h\sqrt{\frac{\alpha t}{k}} \) for which the solution for both cases is approximately the same.

Solution

Step 1
In the sheet “Convection at the surface”, follow the instructions and run the program for various values of \( h\sqrt{\frac{\alpha t}{k}} \) in cell B41.

Step 2
Read the temperature in cell H16.

Step 3
For the case in which the temperature at the surface is constant, run the sheet “Constant surface temperature.”

Step 4
Observe the temperature change in the Temperature vs. Time plot. Observe the difference between the two solutions for low values of \( h\sqrt{\frac{\alpha t}{k}} \). Which is the correct one?

Step 5
Observe the plot of accomplished temperature change \( \frac{T - T_e}{T_o - T_e} \) vs. \( \frac{x}{2\sqrt{\alpha t}} \). Compare it with the respective plot in your textbook, e.g., Ref 3, 4, 5.
Exercise 10.16

Using the spreadsheet program *Semi-Infinite body 3.xls*, calculate the time necessary for the thermal penetration depth to reach the non-heated surface of the steak of Example 10.1. If the temperature of the hotplate changes to 150°C, does the time necessary for the thermal penetration depth to reach the non-heated surface of the steak change? Is the temperature at \( x = 0.02 \) m the same as in the case of 120°C at the time the thermal penetration depth is equal to the thickness of the steak?

**Solution**

**Step 1**
In the sheet “Calculations,” follow the instructions and insert the parameter values. For the heat transfer coefficient use a high value, e.g., 3000W/m²°C in cell B39 (assume a negligible conduct resistance).

**Step 2**
Run the program until Time in cell F4 remains constant. Read the required time in cell G16.

**Step 3**
Observe the temperature change in the Temperature vs. Distance plot.

**Step 4**
Use a hotplate temperature in cell B31 of 150°C. Rerun the program until Time in cell G16 remains constant. Read the required time in cell G16.

**Step 5**
Read the temperature in cell G18 when \( x = 0.02 \) in cell G17. Compare this value with the corresponding temperature when the hot plate temperature is 120°C.

Exercise 10.17

In the “hot-break” process, whole tomatoes are heated with live steam to inactivate the pectin methylesterase enzyme before crushing and extraction, thereby yielding high viscosity products. Calculate the time necessary to heat the center of a tomato 6 cm in diameter to 50°C using 100°C steam if the initial temperature of the tomato is 20°C and the density, the heat capacity, and the thermal conductivity for tomatoes are 1130kg/m³, 3.98kJ/kg°C, and 0.5W/m°C respectively. Find the time necessary for the mass average temperature to reach a value of 82 °C.

**Solution**

Use the spreadsheet *Heat Transfer-Negligible Surface Resistance.xls* to solve the problem. Write the necessary assumptions.
Exercise 10.18

A potato 5 cm in diameter is immersed in boiling water. Calculate the required time for the temperature at the center of the potato to reach 59°C, the onset of starch gelatinization, if the initial temperature of the potato is 21°C and the density, the heat capacity and the thermal conductivity for potatoes are 1120 kg/m³, 3.7 kJ/kg°C and 0.55 W/m°C respectively. Find the mass average temperature of the potato when the center temperature is 59°C. Study the effect of diameter on the required time. Is it proportional to D or to D²?

Solution

Use the spreadsheet *Heat Transfer-Negligible Surface Resistance.xls* to solve the problem. Write the necessary assumptions.